THEORY OF ELASTICITY FOR A SEMILINEAR MATERIAL

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This paper is part of a survey report on nonlinear elasticity theory to the Third All-Union Congress on Theoretical and Applied Mechanics in January 1968. The paper starts with an exposition (Sects. 1–3) of deformation geometry in which representations describing its tensors in terms of their (co- or contravariant) components are avoided. Such a method of describing the phenomena in terms of the quantities (vectors, tensors) giving them directly is conserved throughout the paper. A nonsymmetric Piola tensor is employed to give the state of stress (Sect. 4), and this permits referral of the statics equation and the boundary conditions to the geometry of the initial state of the medium. The specific potential strain energy (Sect. 5) is represented by a function of the invariants expressed in terms of the principal relative elongations; moreover, it is given in the simplest form which reduces to the classical form of linear elasticity theory upon identification of the relative elongations with the diagonal components of the linear strain tensor. John [1] called a material described by such an assignment of the specific potential strain energy "harmonic" since the solution of its plane problems reduces to a (nonlinear) boundary value problem of harmonic function theory. We utilize the designation "semilinear" below ; this can be justified by the fact that the equilibrium differential equations in displacements are linear for such a material in an extensive class of problems, and the nonlinearity is disclosed in the boundary condition.

The solution of problems for a semilinear material is elementary for the simplest equilibrium states characterized by symmetry of the gradient tensors (Sect. 6). The problem of calculating "second order effects" is examined in Sect. 7, a derivation is given of the "second approximation" equation, an example of rod torsion (the evaluation of the elongation under twist) is investigated. Differential equations of equilibrium mode bifurcation are presented in Sect. 8, and are simplified in Sect. 9 under the assumption that the original equilibrium mode will be "simplest" in the mentioned sense. Sensenig [2] gave these equations in another mode of writing. In the particular case when the original equilibrium mode is a homogeneous strain (Sect. 10), we arrive at a system of equations obtained from other (not entirely correct) reasoning of Southwell [3]; a more general representation is given of the solution of this system which is utilized in the problem of bifurcation of the equilibrium of a compressed rod in Sects. 11 and 12. The bifurcation of the equilibrium of a hollow sphere under radially symmetric strain is considered in Sect. 13.

Notation. Material coordinates of points of the medium are denoted by q^s, their Cartesian coordinates in the initial state of the medium (v is the volume bounded by a surface o) by a_s, in the final state (V the volume, and O the surface) by $x_s = a_s + u_s$; under the transformation $v \to V$ the radius vector $\mathbf{r} = \mathbf{i}_s a_s$ becomes $\mathbf{R} = \mathbf{i}_s x_s = \mathbf{r} + \mathbf{u}$.

A vector basis $\mathbf{r}_s = \partial \mathbf{r}/\partial q^s$ is introduced in the v-volume, and the matrix of the covariant components $g_{sk} = \mathbf{r}_s \cdot \mathbf{r}_k$ of the metric tensor $\mathbf{g}$ is defined, where $g = |g_{sk}|$ is its determinant. Contravariant components of this tensor are given by the inverse matrix $\| g^{sk} \|$ so that $g^{sk} g_{kt} = g_t^s$ is the Kronecker symbol. The reciprocal vector basis in the

v-volume is formed by the triple of vectors

$$\mathbf{r}^s = g^{sk}\mathbf{r}_k, \qquad \mathbf{r}^s\cdot\mathbf{r}_k = g_k^{\ s}, \qquad \mathbf{r}^s\cdot\mathbf{r}^k = g^{sk}$$

The same notation, but in capital letters ($\mathbf{R}_s$, $\mathbf{R}^s$, $G_{sk} = \mathbf{R}_s\cdot\mathbf{R}_k$, $G = |G_{sk}|$, etc.) is used in the V-volume.

The unit (metric) tensor $\mathbf{E} = \mathbf{i}_s\mathbf{i}_s$ is represented in the vector bases of the v and V-volumes as

$$\mathbf{E} = \mathbf{g} = g^{sk}\mathbf{r}_s\mathbf{r}_k = g_{sk}\mathbf{r}^s\mathbf{r}^k = \mathbf{r}^s\mathbf{r}_s = \mathbf{r}_s\mathbf{r}^s$$

$$\mathbf{E} = \mathbf{G} = G^{sk}\mathbf{R}_s\mathbf{R}_k = G_{sk}\mathbf{R}^s\mathbf{R}^k = \mathbf{R}^s\mathbf{R}_s = \mathbf{R}_s\mathbf{R}^s$$

Operations in the bases of the V-volume are noted by a prime. The nabla operators in the v and V-volumes are represented, respectively, by the symbolic vectors

$$\nabla = \mathbf{r}^s\frac{\partial}{\partial q^s}, \qquad \nabla' = \mathbf{R}^s\frac{\partial}{\partial q^s}$$

The operation of transposition of a second-rank tensor is indicated by the index T. The density of the medium in the v and V volumes is denoted by ρ_0, ρ and a volume element by $d\tau_0$, $d\tau$, respectively; according to the law of mass conservation $\rho_0\, d\tau_0 = \rho d\tau$.

1. Gradients. Strain measurements. Under the transformation $v \to V$

$$d\mathbf{r} = \mathbf{r}_s dq^s = \mathbf{e}\,|d\mathbf{r}| = \mathbf{e}ds \to d\mathbf{R} = \mathbf{R}_s dq^s = \mathbf{e}'\,|d\mathbf{R}| = \mathbf{e}'dS$$

and taking into account that $dq^s = \mathbf{r}^s\cdot d\mathbf{r} = \mathbf{R}^s\cdot d\mathbf{R}$, we have

$$d\mathbf{R} = \mathbf{R}_s\mathbf{r}^s\cdot d\mathbf{r} = d\mathbf{r}\cdot\mathbf{r}^s\mathbf{R}_s = \nabla\mathbf{R}^T\cdot d\mathbf{r} = d\mathbf{r}\cdot\nabla\mathbf{R}$$
$$d\mathbf{r} = \mathbf{r}_s\mathbf{R}^s\cdot d\mathbf{R} = d\mathbf{R}\cdot\mathbf{R}^s\mathbf{r}_s = \nabla'\mathbf{r}^T\cdot d\mathbf{R} = d\mathbf{R}\cdot\nabla'\mathbf{r} \qquad (1.1)$$

where the tensor-gradients and the transposed gradients

$$\nabla\mathbf{R} = \mathbf{r}^s\mathbf{R}_s, \qquad \nabla'\mathbf{r} = \mathbf{R}^s\mathbf{r}_s, \qquad \nabla\mathbf{R}^T = \mathbf{R}_s\mathbf{r}^s, \qquad \nabla'\mathbf{r}^T = \mathbf{r}_s\mathbf{R}^s \qquad (1.2)$$

have been introduced.

These are mutually inverse tensors

$$\nabla\mathbf{R}\cdot\nabla'\mathbf{r} = \mathbf{E}, \qquad \nabla\mathbf{R}^T\cdot\nabla'\mathbf{r}^T = \mathbf{E} \qquad (1.3)$$

Returning to (1.1) we have

$$\mathbf{e}'dS = \mathbf{e}\cdot\nabla\mathbf{R}ds = \nabla\mathbf{R}^T\cdot\mathbf{e}ds, \qquad \mathbf{e}ds = \mathbf{e}'\cdot\nabla'\mathbf{r}dS = \nabla'\mathbf{r}^T\cdot\mathbf{e}'dS \qquad (1.4)$$

so that

$$dS^2 = \mathbf{e}\cdot\nabla\mathbf{R}\cdot\nabla\mathbf{R}^T\cdot\mathbf{e}ds^2 = \mathbf{e}\cdot\mathbf{G}^\times\cdot\mathbf{e}ds^2, \quad ds^2 = \mathbf{e}'\cdot\nabla'\mathbf{r}\cdot\nabla'\mathbf{r}^T\cdot\mathbf{e}'dS^2 = \mathbf{e}'\cdot\mathbf{g}^\times\cdot\mathbf{e}'dS^2 \qquad (1.5)$$

The measures of the strain

$$\mathbf{G}^\times = \nabla\mathbf{R}\cdot\nabla\mathbf{R}^T = G_{sk}\mathbf{r}^s\mathbf{r}^k, \quad \mathbf{g}^\times = \nabla'\mathbf{r}\cdot\nabla'\mathbf{r}^T = g_{sk}\mathbf{R}^s\mathbf{R}^k \qquad (1.6)$$

have been introduced into the consideration.

They are here defined by their covariant components in the vector bases of the v and V-volumes, which equal the covariant components of the unit tensor (the metric tensors $\mathbf{G}$, $\mathbf{g}$) in the bases of the V- and v-volumes. The inverse tensors are defined by the equalities

$$\mathbf{G}^{\times-1} = (\nabla\mathbf{R}\cdot\nabla\mathbf{R}^T)^{-1} = \nabla'\mathbf{r}^T\cdot\nabla'\mathbf{r} = G^{sk}\mathbf{r}_s\mathbf{r}_k$$
$$\mathbf{g}^{\times-1} = (\nabla'\mathbf{r}\cdot\nabla'\mathbf{r}^T)^{-1} = \nabla\mathbf{R}^T\cdot\nabla\mathbf{R} = g^{sk}\mathbf{R}_s\mathbf{R}_k \qquad (1.7)$$

According to (1.5), (1.4) we have

$$dS = ds\,(\mathbf{e}\cdot\mathbf{G}^\times\cdot\mathbf{e})^{1/2}, \qquad ds = dS\,(\mathbf{e}\cdot\mathbf{g}^\times\cdot\mathbf{e})^{1/2}$$

$$\mathbf{e}' = \frac{\mathbf{e}\cdot\nabla\mathbf{R}}{(\mathbf{e}\cdot\mathbf{G}^\times\cdot\mathbf{e})^{1/2}} = \frac{\nabla\mathbf{R}^T\cdot\mathbf{e}}{(\mathbf{e}\cdot\mathbf{G}^\times\cdot\mathbf{e})^{1/2}}, \qquad \mathbf{e} = \frac{\mathbf{e}'\cdot\nabla'\mathbf{r}}{(\mathbf{e}'\cdot\mathbf{g}^\times\cdot\mathbf{e}')^{1/2}} = \frac{\nabla'\mathbf{r}^T\cdot\mathbf{e}'}{(\mathbf{e}'\cdot\mathbf{g}^\times\cdot\mathbf{e}')^{1/2}} \qquad (1.8)$$

Defining the tensor $G^\times$ by giving its principal values G_s and the orthogonal trihedron of the principal directions e_s, we have

$$G^\times = \sum_s G_s e_s e_s, \qquad G^{\times -1} = \sum_s \frac{e_s e_s}{G_s} \tag{1.9}$$

and by means of (1, 8) we find the expressions of the principal relative elongations δ_k, as well as the unit vectors e_k' into which the e_k transform under the transformation $v \to V$

$$\delta_k = \frac{dS_k - ds_k}{ds_k} = \sqrt{G_k} - 1, \qquad e_k' = \frac{e_k \cdot \nabla R}{\sqrt{G_k}} = \frac{\nabla R^T \cdot e_k}{\sqrt{G_k}} \tag{1.10}$$

and it is easy to verify that the trihedron e_k' is orthogonal. This is the trihedron of the principal directions of the tensors $g^{\times -1}$, $g^\times$. Indeed, we have according to (1, 7)–(1, 10)

$$\sum G_s e_s' e_s' = \nabla R^T \cdot e_s e_s \cdot \nabla R = \nabla R^T \cdot E \cdot \nabla R = \nabla R^T \cdot \nabla R = g^{\times -1}$$

Thus

$$g^\times = \sum_s \frac{e_s' e_s'}{G_s}, \qquad g^{\times -1} = \sum G_s e_s' e_s' \tag{1.11}$$

and it is thereby shown that the principal values of the pairs of tensors $G^\times$, $g^{\times -1}$ and $G^{\times -1}$, $g^\times$ are equal; however, this follows from the equality of the eigenvalues of the matrices AB and BA (see (1, 6), (1, 7)).

The transformation $v \to V$ is therefore connected with the rotation of the trihedron $e_s \to e_s'$; under the inverse transformation $e_s' \to e_s$ and according to (1, 8), (1, 11)

$$e_s = \sqrt{G_s} e_s' \cdot \nabla' r = \sqrt{G_s} \nabla' r^T \cdot e_s' \tag{1.12}$$

But according to (1, 10) we also have

$$\nabla R \cdot e_s' = \frac{1}{\sqrt{G_s}} \nabla R \cdot \nabla R^T \cdot e_s = \frac{1}{\sqrt{G_s}} G^\times \cdot e_s = \sqrt{G_s} e_s$$

so that formulas inverse to (1, 10) can be written thus

$$e_s = \frac{\nabla R \cdot e_s'}{\sqrt{G_s}} = \frac{e_s' \cdot \nabla R^T}{\sqrt{G_s}} \tag{1.13}$$

2. Rotation tensors. The tensors

$$A = \sum_s e_s e_s', \qquad A^T = \sum_s e_s' e_s \tag{2.1}$$

are introduced so that

$$A \cdot A^T = \sum_{s,k} e_s e_s' \cdot e_k' e_k = \sum_s e_s e_s = E, \qquad A^T = A^{-1} \tag{2.2}$$

The tensor inverse to the transpose is a rotation tensor; it rotates the trihedron e_s into e_s

$$e_s' = e_s \cdot A = A^T \cdot e_s, \qquad e_s = e_s' \cdot A^T = A \cdot e_s' \tag{2.3}$$

Returning to (1, 10) and introducing the tensors

$$G^{\times 1/2} = \sum_s \sqrt{G_s} e_s e_s, \qquad G^{\times(-1/2)} = \sum_s \frac{e_s e_s}{\sqrt{G_s}} \tag{2.4}$$

the rotation tensors can be represented as

$$\begin{gathered} A = \sum_s e_s e_s' = \sum_s \frac{e_s e_s}{\sqrt{G_s}} \cdot \nabla R = G^{\times(-1/2)} \cdot \nabla R, \quad A^T = \nabla R^T \cdot G^{\times(-1/2)} \\ A = (A^T)^{-1} = G^{\times 1/2} \cdot \nabla' r^T, \qquad A^T = A^{-1} = \nabla' r \cdot G^{\times 1/2} \end{gathered} \tag{2.5}$$

Representations of the gradients as

$$\nabla \mathbf{R} = \mathbf{G}^{\times 1/2}\cdot \mathbf{A}, \quad \nabla \mathbf{R}^T = \mathbf{A}^T\cdot \mathbf{G}^{\times 1/2}; \qquad \nabla \mathbf{r} = \mathbf{A}^T\cdot \mathbf{G}^{\times(-1/2)}, \quad \nabla \mathbf{r}^T = \mathbf{G}^{\times(-1/2)}\cdot \mathbf{A} \tag{2.6}$$

will be a consequence of these formulas.

The relationships

$$\mathbf{A}^T\cdot \mathbf{G}^{\times}\cdot \mathbf{A} = \mathbf{g}^{\times -1}, \quad \mathbf{G}^{\times} = \mathbf{A}\cdot \mathbf{g}^{\times -1}\cdot \mathbf{A}^T; \qquad \mathbf{g}^{\times} = \mathbf{A}^T\cdot \mathbf{G}^{\times -1}\cdot A, \quad \mathbf{G}^{\times -1} = \mathbf{A}\cdot \mathbf{g}^{\times}\cdot \mathbf{A}^T \tag{2.7}$$

are also verified directly.

3. Directed area. Under the transformation $v \to V$

$$d\tau_0 = r_1\cdot(\mathbf{r}_2\times \mathbf{r}_3)\, dq^1dq^2dq^3 = \sqrt{g}\,dq^1dq^2dq^3$$

$$d\tau = \mathbf{R}_1\cdot(\mathbf{R}_2\times \mathbf{R}_3)\,dq^1dq^2dq^3 = \sqrt{G}\,dq^1dq^2dq^3$$

Moreover, considering the elementary parallelepiped with edges δs_k directed along the principal axes $\mathbf{e}_k$ of the tensor $\mathbf{G}^{\times}$, we have according to (1.10)

$$d\tau_0 = \delta s_1\delta s_2\delta s_3 \to d\tau = \sqrt{G_1G_2G_3}\,\delta s_1\delta s_2\delta s_3 = \sqrt{G_1G_2G_3}\,d\tau_0$$

Thus

$$d\tau / d\tau_0 = \sqrt{G/g} = \sqrt{G_1G_2G_3} \tag{3.1}$$

In the elementary tetrahedron $OA_1A_2A_3$ with edges $\overrightarrow{OA}_k = \mathbf{e}_k\delta s_k$ directed along the same axes, the normal vector $\mathbf{n}do$ Of the area $A_1A_2A_3$ directed out of the tetrahedron is

$$2\mathbf{n}do = \mathbf{e}_2\times \mathbf{e}_3\delta s_2\delta s_3 + \mathbf{e}_3\times \mathbf{e}_1\delta s_3\delta s_1 + \mathbf{e}_1\times \mathbf{e}_2\delta s_1\delta s_2 = d\tau_0 \sum_k \frac{\mathbf{e}_k}{\delta s_k}$$

Under the transformation $v \to V$, this vector is represented as (see (3.1),(1.10))

$$2\mathbf{N}dO = d\tau \sum_k \frac{\mathbf{e}_k'}{\delta S_k} = d\tau_0\left(\frac{G}{g}\right)^{1/2} \nabla \mathbf{R}^T\cdot \sum_k \frac{\mathbf{e}_k}{G_k\delta s_k} = d\tau_0\left(\frac{G}{g}\right)^{1/2} \nabla \mathbf{R}^T\cdot \mathbf{G}^{\times -1}\cdot \sum_k \frac{\mathbf{e}_k}{\delta s_k}$$

We then arrive at the relationship used repeatedly later (see (1.7),(1.3))

$$\mathbf{N}\, dO = \sqrt{G/g}\,\nabla \mathbf{R}^T\cdot \mathbf{G}^{\times -1}\cdot \mathbf{n}do = \sqrt{G/g}\,\nabla \mathbf{r}\cdot \mathbf{n}do = \sqrt{G/g}\,\mathbf{n}\cdot \nabla \mathbf{r}^T do \tag{3.2}$$

We thence also have

$$\frac{dO}{do} = \left(\frac{G}{g}\,\mathbf{n}\cdot\nabla \mathbf{r}^T\cdot \nabla \mathbf{r}\cdot \mathbf{n}\right)^{1/2} = \left(\frac{G}{g}\,\mathbf{n}\cdot \mathbf{G}^{\times -1}\cdot \mathbf{n}\right)^{1/2} \tag{3.3}$$

4. Stress tensor. The symmetric second-rank tensor $\mathbf{T} = \mathbf{T}^T$ given in the volume V is a stress tensor if its product by the vector $\mathbf{N}dO$ of the directed area determines the force $\mathbf{F}dO$ acting on this area.

According to this definition, we have by referring to (3.2)

$$\mathbf{F}dO = \mathbf{N}\cdot \mathbf{T}\, dO = \sqrt{G/g}\,\mathbf{n}\cdot \nabla \mathbf{r}^T\cdot \mathbf{T}do \tag{4.1}$$

and this relationship indicates the expediency of introducing a nonsymmetric tensor $\mathbf{D}$, the Piola stress tensor (1831)

$$\mathbf{D} \doteq \sqrt{G/g}\,\nabla \mathbf{r}^T\cdot \mathbf{T}, \qquad \mathbf{D}^T = \sqrt{G/g}\,\mathbf{T}\cdot \nabla \mathbf{r} \tag{4.2}$$

Thus

$$\mathbf{F}dO = \mathbf{n}\cdot \mathbf{D}do, \qquad \mathbf{F}\sqrt{G/g}\,(\mathbf{n}\cdot \mathbf{G}^{\times -1}\cdot \mathbf{n})^{1/2} = \mathbf{n}\cdot \mathbf{D} \tag{4.3}$$

Referring to (2.6), we have

$$\mathbf{D}\cdot \mathbf{A}^T = \sqrt{G/g}\,\mathbf{G}^{\times(-1/2)}\cdot \mathbf{A}\cdot \mathbf{T}\cdot \mathbf{A}^T, \quad \mathbf{A}\cdot \mathbf{D}^T = \sqrt{G/g}\,\mathbf{A}\cdot \mathbf{T}\cdot \mathbf{A}^T\cdot \mathbf{G}^{\times(-1/2)} \tag{4.4}$$

It is known that the tensors and $\mathbf{g}^{\times}$(in an isotropic medium) are coaxial ($\mathbf{e}_s'$ is the trihedron of their principal directions); hence, the "rotated stress tensor" $\mathbf{A}\cdot\mathbf{T}\cdot\mathbf{A}^T$ is coaxial with $\mathbf{G}^{\times}$, which means that the right sides of (4.4) are equal; the symmetry of the tensor $\mathbf{D}\cdot\mathbf{A}^T$ hence follows

$$\mathbf{D}\cdot\mathbf{A}^T = \mathbf{A}\cdot\mathbf{D}^T = (\mathbf{D}\cdot\mathbf{A}^T)^T$$

The condition for the principal vector of the forces acting on a volume V_* bounded by the surface O_* separated arbitrarily out of the volume V into which the r_*-volume (bounded by the surface o_*) is transformed by $v \to V$, to vanish is written as

$$\iint\limits_{O_*} \mathbf{F}dO + \iiint\limits_{V_*} \rho\mathbf{K}d\tau = \iint\limits_{o_*} \mathbf{n}\cdot\mathbf{D}do + \iiint\limits_{v_*} \rho_0\mathbf{K}d\tau_0 = 0$$

We arrive at the necessary equilibrium condition

$$\iiint\limits_{r_*} (\nabla\cdot\mathbf{D}+\rho_0\mathbf{K})\, d\tau_0 = 0, \quad \text{or} \quad \nabla\cdot\mathbf{D} + \rho_0\mathbf{K} = 0 \qquad (4.5)$$

where $\mathbf{K}$ is the mass force vector. The condition for the principal moment of the forces $\mathbf{F}dO$, $\rho\mathbf{K}$ to vanish finally reduces to the requirement that the tensor $\mathbf{T}$ be symmetric.

5. Equation of state. Referring to (2.5), (2.6), we represent (4.4) as

$$\mathbf{D}\cdot\mathbf{A}^T = \sqrt{G/g}\,\nabla\mathbf{r}^T\cdot\mathbf{T}\cdot\nabla\mathbf{r}\cdot\mathbf{G}^{\times 1/2} \qquad (5.1)$$

It is moreover known that the variation in specific potential strain energy of an elastic solid is represented as the trace, the semiproduct of the stress energy tensor $\mathbf{Q} = \nabla'\mathbf{r}^T\cdot\mathbf{T}\cdot\nabla'\mathbf{r}$ and the tensor $\delta\mathbf{G}^\times$

$$\delta W = {}^1/_2\, \sqrt{G/g}\,\nabla\mathbf{r}^T\cdot\mathbf{T}\cdot\nabla\mathbf{r}\cdot\cdot\delta\mathbf{G}^\times \qquad (5.2)$$

Hence, it is expressed in terms of the Piola tensor as

$$\delta W = {}^1/_2\mathbf{D}\cdot\mathbf{A}^T\cdot\mathbf{G}^{\times(-1/2)}\cdot\cdot\delta\mathbf{G}^\times = \mathbf{D}\cdot\mathbf{A}^T\cdot\cdot{}^1/_2\mathbf{G}^{\times(-1/2)}\cdot\delta\mathbf{G}^\times$$

or

$$\delta W = \mathbf{D}\cdot\mathbf{A}^T\cdot\cdot\delta\mathbf{G}^{\times 1/2} \qquad (5.3)$$

This energy is later considered as a function of three invariants of the state of strain

$$s_k = \delta_1{}^k + \delta_2{}^k + \delta_3{}^k \qquad (k = 1, 2, 3) \qquad (5.4)$$

where δ_s are the principal relative elongations (see (1.10)). We obtain

$$s_1 = I_1(\mathbf{G}^{\times 1/2}) - 3, \quad s_2 = (\sqrt{G_1}-1)^2 + (\sqrt{G_2}-1)^2 + (\sqrt{G_3}-1)^2 = I_1(\mathbf{G}^\times) - 2I_1(\mathbf{G}^{\times 1/2}) + 3$$

$$s_3 = (\sqrt{G_1}-1)^3 + (\sqrt{G_2}-1)^3 + (\sqrt{G_3}-1)^3 = I_1(\mathbf{G}^{\times 3/2}) - 3I_1(\mathbf{G}^\times) + 3I_1(\mathbf{G}^{\times 1/2}) - 3$$

where $I_1(\mathbf{Q})$ is the first invariant of the tensor Q. We hence find

$$\delta W = \frac{\partial W}{\partial s_1}\delta s_1 + \frac{\partial W}{\partial s_2}\delta s_2 + \frac{\partial W}{\partial s_3}\delta s_3 = \left(\frac{\partial W}{\partial s_1} - 2\frac{\partial W}{\partial s_2} + 3\frac{\partial W}{\partial s_3}\right)\delta I_1(\mathbf{G}^{\times 1/2}) +$$
$$+ \left(\frac{\partial W}{\partial s_2} - 3\frac{\partial W}{\partial s_3}\right)\delta I_1(\mathbf{G}^\times) + \frac{\partial W}{\partial s_3}\delta I_1(\mathbf{G}^{\times 3/2}) \qquad (5.5)$$

We now represent the symmetric tensor $\mathbf{D}\cdot\mathbf{A}^T$ by the quadratic form of the tensor $\mathbf{G}^{\times 1/2}$

$$\mathbf{D}\cdot\mathbf{A}^T = a\mathbf{E} + b\mathbf{G}^{\times 1/2} + c\mathbf{G}^\times$$

The scalar multipliers are here functions of the invariants s_k; we have

$$\mathbf{E}\cdot\cdot\delta\mathbf{G}^{\times 1/2} = I_1(\delta\mathbf{G}^{\times 1/2}) = \delta I_1(\mathbf{G}^{\times 1/2})$$
$$\mathbf{G}^{\times 1/2}\cdot\cdot\delta\mathbf{G}^{\times 1/2} = I_1(\mathbf{G}^{\times 1/2}\cdot\delta\mathbf{G}^{\times 1/2}) = {}^1/_2\delta I_1(\mathbf{G}^\times)$$
$$\mathbf{G}^\times\cdot\cdot\delta\mathbf{G}^{\times 1/2} = I_1(\mathbf{G}^\times\cdot\delta\mathbf{G}^{\times 1/2}) = {}^1/_3\delta I_1(\mathbf{G}^{\times 3/2})$$

and according to (5.3), (5.5), (2.6) we arrive at the equation of state

$$\mathbf{D} = \left(\frac{\partial W}{\partial s_1} - 2\frac{\partial W}{\partial s_2} + 3\frac{\partial W}{\partial s_3}\right)\mathbf{A} + 2\left(\frac{\partial W}{\partial s_2} - 3\frac{\partial W}{\partial s_3}\right)\nabla\mathbf{R} + 3\frac{\partial W}{\partial s_3}\mathbf{G}^{\times 1/2}\cdot\nabla\mathbf{R} \qquad (5.6)$$

which connects the stress with quantities defined by deformations of the v-volume into the V-volume. John proposed the expression of the specific potential energy as

$$W = {}^1/_2\ \lambda s_1^2 + \mu s_2 \tag{5.7}$$

which agrees outwardly with the way it is given in linear elasticity theory when s_1, s_2 are the first invariants of the linear strain tensor $\mathbf{s}$ and its square. A material subject to this law is called by John [1] "harmonic" material. Here it is called "semilinear". According to the above

$$\mathbf{D} = (\lambda s_1 - 2\mu)\,\mathbf{A} + 2\mu \nabla \mathbf{R} \tag{5.8}$$

and substituted into (4.5) results in an analog to the "equilibrium equations in displacements"

$$(\lambda s_1 - 2\mu)\nabla\cdot\mathbf{A} + \lambda\mathbf{A}^T\cdot\nabla s_1 + 2\mu\nabla^2\mathbf{R} + \rho_0\mathbf{K} = 0 \tag{5.9}$$

Upon assigning the surface forces $\mathbf{F}$ the boundary condition (4.3) is written as

$$\mathbf{F}\frac{dO}{do} = (\lambda s_1 - 2\mu)\,\mathbf{n}\cdot\mathbf{A} + 2\mu\mathbf{n}\cdot\nabla\mathbf{R} \tag{5.10}$$

Written thus it presumes knowledge of the shape of the body boundary (the vector $\mathbf{n}$) in the initial state, and the equilibrium equations (5.8) are referred to the vector basis of the same state. This is the advantage of using the Piola tensor. To apply the equation of state (5.6), or particularly (5.7), the expressions of the principal values and principal directions of the tensor $\mathbf{G}^\times$ by means of assigning the point transformation $v \to V$ must be known.

It should here be recalled that the process of solving nonlinear elasticity theory problems reduces, as a rule, to assigning the point transformation $v \to V$ and then seeking the distribution of the surface forces assuring maintenance of this strain state of the solid. In all cases which result without exception in closed solutions, the transformation is given in simplest form when rotation of the principal axes is either absent ($\mathbf{A} = \mathbf{E}$) or retains a constant value over the whole solid ($\mathbf{A}$ is a constant tensor). Illustrations are: axisymmetric strain of a circular cylinder, radially symmetric strain of the sphere, triaxial uniform tension ($\mathbf{A} = \mathbf{E}$), affine transformation, and in particular, simple shear strain ($\mathbf{A}$ is a constant tensor). It must be added that closed solutions are usually obtained successfully only for an incompressible Mooney material.

Under the listed assumptions, the differential equations for a "harmonic" material turn out to be linear, and the nonlinearity of the problem is produced by the boundary conditions.

6. Conservation of principal directions. This will hold if the tensor-gradients are symmetric

$$\nabla\mathbf{R} = \nabla\mathbf{R}^T, \qquad \nabla\mathbf{r} = \nabla\mathbf{r}^T \tag{6.1}$$

Then according to (1.6), (2.6), (2.1)

$$\nabla\mathbf{R} = \nabla\mathbf{R}^T = \mathbf{G}^{\times 1/2}, \qquad \mathbf{A} = \mathbf{E}, \qquad \mathbf{e}_s = \mathbf{e}_s' \tag{6.2}$$

For example, the vectors $\mathbf{e}_s$ ($\mathbf{e}_s'$) coincide with the unit basis vectors of an orthogonal (cylindrical, spherical) coordinate system in the axisymmetric strain of a circular cylinder and radially symmetric strain of a sphere. We now have

$$s_1 = I_1(\mathbf{G}^{\times 1/2}) - 3 = I_1(\nabla\mathbf{R}) - 3 = \vartheta = \nabla\cdot\mathbf{u}$$

and Eqs. (5.8), (5.9) are represented as

$$(\lambda + 2\mu)\nabla^2\mathbf{u} + \rho_0\mathbf{K} = 0, \qquad \lambda\mathbf{n}\nabla\cdot\mathbf{u} + 2\mu\mathbf{n}\cdot\nabla\mathbf{u} = \mathbf{F}\frac{dO}{do} \tag{6.3}$$

since $\nabla\mathbf{u} = \nabla\mathbf{u}^T$, $\nabla^2\mathbf{u} = \nabla\nabla\cdot\mathbf{u}$ according to (6.1). Under such conditions these equations are the simplest particular form of the elasticity theory equations in displacements; only the right side of the boundary condition is nonlinear. In the absence of mass forces

$\mathbf{u}$ is a harmonic vector. An illustration is the vector

$$\mathbf{u} = C_1\mathbf{R} + \frac{C_2}{R^3}\mathbf{R} = C_1\mathbf{R} - C_2\nabla\frac{1}{R} = \left(C_1 R + \frac{C_2}{R^2}\right)\mathbf{e}_R \tag{6.4}$$

governing the radially symmetric strain, in which the surface of any sphere of radius R becomes a sphere of radius

$$f(R) = (C_1 + 1)R + C_2/R^2 \tag{6.5}$$

This strain is realizable under the effect of uniformly distributed pressures p_1, p_0 on the inner ($R = R_1$) and outer ($R = R_0$) surfaces of the sphere

$$R = R_0 : \mathbf{F}\frac{dO}{do} = -p_0\mathbf{e}_R\frac{f^2(R_0)}{R_0^2}, \qquad R = R_1 : \mathbf{F}\frac{dO}{do} = p_1\mathbf{e}_R\frac{f^2(R_1)}{R_1^2}$$

The constants C_1, C_2 are determined by means of the boundary conditions (6.3). We have

$$\nabla\mathbf{u} = C_1\mathbf{E} + C_2\left(\frac{\mathbf{E}}{R^3} - 3\frac{\mathbf{RR}}{R^5}\right), \qquad \nabla\cdot\mathbf{u} = 3C_1 \tag{6.6}$$

and moreover

$$-p_0\frac{f^2(R_0)}{R_0^2} = (3\lambda + 2\mu)C_1 - \frac{4\mu}{R_0^3}C_2, \qquad p_1\frac{f^2(R_1)}{R_1^2} = (3\lambda + 2\mu)C_1 - \frac{4\mu}{R_1^3}C_2$$

For $p_1 = 0$, $p_0 = p$ for example (external pressure)

$$p_0 = -\frac{(3\lambda + 2\mu)(1 - \varkappa)C_1}{[C_1 + 1 + \varkappa C_1(3\lambda + 2\mu)/4\mu]^2} \qquad \left(\varkappa = \left(\frac{R_1}{R_0}\right)^3\right) \tag{6.7}$$

and C_1 is the root of this quadratic equation (greater than -1).

7. Second order effects. The measure of the strain (1.6) is represented as

$$\mathbf{G}^\times = \nabla\mathbf{R}\cdot\nabla\mathbf{R}^T = (\mathbf{E} + \nabla\mathbf{u})\cdot(\mathbf{E} + \nabla\mathbf{u}^T) = \mathbf{E} + 2\boldsymbol{\varepsilon} + \nabla\mathbf{u}\cdot\nabla\mathbf{u}^T$$

where $\boldsymbol{\varepsilon}$ is the linear strain tensor, and the tensors $\nabla\mathbf{u}$, $\nabla\mathbf{u}^T$ are representable by their partitions into symmetric ($\boldsymbol{\varepsilon}$) and skew-symmetric ($\boldsymbol{\Omega}$) parts

$$\boldsymbol{\varepsilon} = {}^1/_2(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \qquad \nabla\mathbf{u} = \boldsymbol{\varepsilon} - \boldsymbol{\Omega}, \qquad \Delta\mathbf{u}^T = \boldsymbol{\varepsilon} + \boldsymbol{\Omega}$$

where the tensor $\boldsymbol{\Omega}$ can be expressed in terms of the linear rotation vector $\boldsymbol{\omega}$

$$\boldsymbol{\Omega} = \mathbf{E}\times\boldsymbol{\omega} = \boldsymbol{\omega}\times\mathbf{E}, \qquad \boldsymbol{\omega} = {}^1/_2\nabla\times\mathbf{u}$$

Hence

$$\mathbf{G}^\times = \mathbf{E} + 2\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^2 - \boldsymbol{\Omega}^2 + \boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\varepsilon}$$

and moreover, by retaining just the component of second degree in derivatives of the vector $\mathbf{u}$

$$\mathbf{G}^{\times 1/2} = \mathbf{E} + \boldsymbol{\varepsilon} - {}^1/_2(\boldsymbol{\Omega}^2 - \boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega} + \boldsymbol{\Omega}\cdot\boldsymbol{\varepsilon})$$

$$\mathbf{G}^{\times(-1/2)} = \mathbf{E} - \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^2 + {}^1/_2(\boldsymbol{\Omega}^2 - \mathbf{E}\cdot\boldsymbol{\varepsilon} + \boldsymbol{\Omega}\cdot\boldsymbol{\varepsilon})$$

Hence, to the same accuracy as in (2.5), we have

$$\varepsilon_1 = I(\mathbf{G}^{\times 1/2}) - 3 = I_1(\boldsymbol{\varepsilon}) - {}^1/_2(\boldsymbol{\Omega}\cdot\cdot\boldsymbol{\Omega} - \boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{\Omega} + \boldsymbol{\Omega}\cdot\cdot\boldsymbol{\varepsilon}) = \vartheta + \boldsymbol{\omega}\cdot\boldsymbol{\omega} \tag{7.1}$$

$$\mathbf{A} = \mathbf{G}^{\times(-1/2)}\cdot\nabla\mathbf{R} = \mathbf{G}^{\times(-1/2)}\cdot(\mathbf{E} + \boldsymbol{\varepsilon} - \boldsymbol{\Omega}) = \mathbf{E} - \boldsymbol{\Omega} + {}^1/_2(\boldsymbol{\Omega}^2 + \boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega} + \boldsymbol{\Omega}\cdot\boldsymbol{\varepsilon}) \tag{7.2}$$

and, according to (5.8), we arrive at the following representation of the Piola tensor

$$\mathbf{D} = \lambda\vartheta\mathbf{E} + 2\mu\boldsymbol{\varepsilon} - (\lambda\vartheta\mathbf{E} + 2\mu\boldsymbol{\varepsilon})\cdot\boldsymbol{\Omega} + (\lambda + \mu)\boldsymbol{\omega}\cdot\boldsymbol{\omega}\mathbf{E} - \mu\boldsymbol{\omega}\boldsymbol{\omega} + \mu(\boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\varepsilon})$$

or otherwise

$$\mathbf{D} = \mathbf{T}^\circ(\mathbf{u}) - \mathbf{T}^\circ(\mathbf{u})\times\boldsymbol{\omega} + \mathbf{T}^*(\mathbf{u}) \tag{7.3}$$

Here $\mathbf{T}^c(\mathbf{u})$ is the stress tensor of linear elasticity theory, evaluated by means of the vector $\mathbf{u}$ according to Hooke's law, $\mathbf{T}^*(\mathbf{u})$ is an additional symmetric tensor

$$\mathbf{T}^\circ(\mathbf{u}) = \lambda\vartheta\mathbf{E} + 2\mu\boldsymbol{\varepsilon}, \quad \mathbf{T}^*(\mathbf{u}) = (\lambda + \mu)\boldsymbol{\omega}\cdot\boldsymbol{\omega}\mathbf{E} - \mu\boldsymbol{\omega}\boldsymbol{\omega} +$$
$$+ \mu(\boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\varepsilon}) = \lambda\boldsymbol{\omega}\cdot\boldsymbol{\omega}\mathbf{E} - \mu\boldsymbol{\varepsilon}^2 + \mu\nabla\mathbf{u}\cdot\nabla\mathbf{u}^T \tag{7.4}$$

The skew-symmetric part of the tensor $\mathbf{D}$ is represented by the tensor $-\mathbf{T}^\circ(\mathbf{u})\times\boldsymbol{\omega}$.

The equilibrium equations in the volume and on the surface are now written according to (4.3), (4.5) as

$$\nabla\cdot\mathbf{T}^{\bullet}(\mathbf{u})+\boldsymbol{\omega}\times\nabla\cdot\mathbf{T}^{\circ}(\mathbf{u})-[\mathbf{T}^{\bullet}(\mathbf{u})\cdot\nabla]\times\boldsymbol{\omega}+\nabla\cdot\mathbf{T}^{\bullet}(\mathbf{u})=0 \quad (7.5)$$

$$\mathbf{F}\frac{dO}{do}=\mathbf{n}\cdot\mathbf{T}^{\bullet}(\mathbf{u})+\boldsymbol{\omega}\times[\mathbf{n}\cdot\mathbf{T}^{\bullet}(\mathbf{u})]+\mathbf{n}\cdot\mathbf{T}^{\bullet}(\mathbf{u}) \quad (7.6)$$

(volume forces are assumed absent).

Now assuming $\mathbf{u}=\mathbf{v}+\mathbf{w}$, we determine the vector $\mathbf{v}$ as the solution of the linear problem

$$\nabla\cdot\mathbf{T}^{\bullet}(\mathbf{v})=0, \qquad \mathbf{n}\cdot\mathbf{T}^{\circ}(\mathbf{v})=\mathbf{F}\frac{dO}{do}=\mathbf{F}^{\bullet} \quad (7.7)$$

Here $\mathbf{F}^{\circ}$ is the surface force referred to unit area of the surface $\mathbf{o}$ bounding the volume v. The problem has a solution since the external forces are assumed statically equivalent to zero in the sequence of equilibrium states in the transition from the volume v to the volume V

$$\iint\limits_{O}\mathbf{F}dO=\iint\limits_{o}\mathbf{F}^{\circ}do=0, \qquad \iint\limits_{O}\mathbf{R}\times\mathbf{F}dO=\iint\limits_{o}\mathbf{r}\times\mathbf{F}^{\bullet}do=0 \quad (7.8)$$

The vector $\mathbf{w}$ defining the desired "second order effect", is given, according to (7.5), (7.6), by the solution of the problem of linear theory

$$\nabla\cdot\mathbf{T}^{\circ}(\mathbf{w})+\mathbf{k}=0, \qquad \mathbf{n}\cdot\mathbf{T}^{\bullet}(\mathbf{w})=\mathbf{f} \quad (7.9)$$

with the "volume and surface forces"

$$\mathbf{k}=-[\mathbf{T}^{\bullet}(\mathbf{v})\cdot\nabla]\times\boldsymbol{\omega}+\nabla\cdot\mathbf{T}^{\bullet}(\mathbf{v}), \qquad \mathbf{f}=-\boldsymbol{\omega}\times\mathbf{F}^{\circ}-\mathbf{n}\cdot\mathbf{T}^{\bullet}(\mathbf{v}) \quad (7.10)$$

where Eqs. (7.7) are taken into account.

The problem has a solution if the principal vector and principal moment are zero

$$\iint\limits_{o}\mathbf{f}do+\iiint\limits_{v}\mathbf{k}d\tau_0=0, \quad \iint\limits_{o}\mathbf{r}\times\mathbf{f}do+\iiint\limits_{v}\mathbf{r}\times\mathbf{k}d\tau_0=0 \quad (7.11)$$

Compliance with the first condition is easily verified. This follows from the identities already utilized earlier

$$\boldsymbol{\omega}\times(\mathbf{n}\cdot\mathbf{T}^{\bullet})=-\mathbf{n}\cdot(\mathbf{T}^{\bullet}\times\boldsymbol{\omega}), \qquad \nabla\cdot(\mathbf{T}^{\circ}\times\boldsymbol{\omega})=-\boldsymbol{\omega}\times\nabla\cdot\mathbf{T}^{\circ}+(\mathbf{T}^{\bullet}\cdot\nabla)\times\boldsymbol{\omega} \quad (7.12)$$

which are valid for the symmetric tensor $\mathbf{T}^{\bullet}$. Hence, referring to (7.7), (7.10), we have

$$\iint\limits_{o}\mathbf{f}do=\iiint\limits_{v}[\nabla\cdot(\mathbf{T}^{\circ}\times\boldsymbol{\omega})-\nabla\cdot\mathbf{T}^{\bullet}]\,d\tau_0=\iiint\limits_{v}[(\mathbf{T}^{\bullet}\cdot\nabla)\times\boldsymbol{\omega}-\nabla\cdot\mathbf{T}^{\bullet}]\,d\tau_0=-\iiint\limits_{v}\mathbf{k}d\tau_0$$

q.e.d. The situation with the second condition in (7.11) is different. It is here necessary to refer to the relationship

$$\iint\limits_{o}\mathbf{r}\times\mathbf{n}\cdot Q do=-\iiint\limits_{o}\mathbf{n}\cdot(Q\times\mathbf{r})\,do=-\iiint\limits_{v}\nabla\cdot(Q\times\mathbf{r})\,d\tau_0=$$

$$=\iiint\limits_{v}\mathbf{r}\times\nabla\cdot Q d\tau_0+\iiint\limits_{v}\mathbf{i}_s\times(\mathbf{i}_s\cdot Q)\,d\tau_0$$

where the last member drops out if Q is a symmetric tensor. As applied to the vector $\mathbf{f}$ we have

$$Q=\mathbf{T}^{\bullet}(\mathbf{v})\times\boldsymbol{\omega}-\mathbf{T}^{\bullet}(\mathbf{v}) \quad (7.13)$$

and referring to (7.10), (7.12), we arrive at the condition

$$\iint\limits_{O}\mathbf{r}\times\mathbf{f}dO+\iiint\limits_{v}\mathbf{r}\times\mathbf{k}d\tau_0=\iiint\limits_{v}\mathbf{i}_s\times[\mathbf{i}_s\cdot(\mathrm{T}\times\boldsymbol{\omega})]\,d\tau_0=0$$

The vector under the integral sign can be represented in the invariant form

$$\mathbf{i}_s\times[\mathbf{i}_s\cdot(\mathbf{T}^{\circ}\times\boldsymbol{\omega})]=\boldsymbol{\omega}\cdot\mathbf{T}^{\bullet}-\boldsymbol{\omega}I_1(\mathbf{T}^{\circ})$$

so that the second condition of (7.11) will result in the requirement

$$\iiint_{\tau_2} [\boldsymbol{\omega}\cdot \mathbf{T}^\circ(\mathbf{v}) - \boldsymbol{\omega} I_1(\mathbf{T}^\circ)]\, d\tau_0 = 0 \tag{7.14}$$

Let us note that the vector $\boldsymbol{\omega}$ is determined by the solution of the linear boundary value problem (7.7) to the accuracy of an additive constant vector $\boldsymbol{\omega}_0$ so that $\boldsymbol{\omega} = \boldsymbol{\omega}' + \boldsymbol{\omega}_0$, where $\boldsymbol{\omega}'(0, 0, 0) = 0$, say. The vector $\boldsymbol{\omega}_0$ can be subjected to the condition

$$\boldsymbol{\omega}_0 \cdot \iiint_v [\mathbf{T}^\circ(\mathbf{v}) - \mathbf{E} I_1(\mathbf{T}^\circ(\mathbf{v}))]\, d\tau_0 = v\mathbf{b}, \qquad \mathbf{b} = -\frac{1}{v}\iiint_v [\boldsymbol{\omega}'\cdot \mathbf{T}^\circ - \boldsymbol{\omega}' I_1(\mathbf{T}^\circ)]\, d\tau_0$$

in which $\mathbf{b}$ is a vector evaluated by the solution of the linear problem (7.7). We have arrived at a system of linear equations for the unknown vector $\boldsymbol{\omega}_0$

$$\omega_{0r}(c_{rq} - c\delta_{rq}) = b_q, \qquad c_{rq} = \frac{1}{v}\iiint_v t_{rq}^\circ\, d\tau_0, \qquad c = c_{11} + c_{22} + c_{33}$$

for which the coefficients on the left side are the mean values of the components of the tensor $\mathbf{T}^\circ$ $(\mathbf{v})$ expressed in terms of the surface forces, but their determination does not require the solution o the boundary value problem (7.7). The determinant of this system should be nonzero

$$\Delta = |c_{rq} - c\delta_{rq}| \neq 0 \tag{7.15}$$

and when this condition is not satisfied (for $\Delta = 0$) the boundary value problem (7.9) also cannot have a solution. Taking account of the nonlinearity effect is not achieved by inserting the correction $\mathbf{w}$ into the solution of the linear problem.

Seeking the vector $\mathbf{w}$ is certainly made difficult by the complexity of the assignment (7.10) of the volume and surface forces in the boundary value problem (7.9). Application of the reciprocity theorem permits the determination of the mean values of the strains determined by the vector $\mathbf{w}$ in terms of these forces; thus taking account of the nonlinearity effect can be satisfied for the calculation of the integral effects, the changes in length, volume, etc., when the need to take account of the influence of nonlinearity on the stress distribution is removed to the second stage. The calculation dictated by the reciprocity theorem is simplified somewhat because of the special structure of the vectors $\mathbf{k}$, $\mathbf{f}$. Application of this theorem results in the relationship

$$\frac{\nu}{1-2\nu}\,\vartheta'\vartheta_m(\mathbf{w}) + \boldsymbol{\varepsilon}'\cdot\cdot\,\boldsymbol{\varepsilon}_m(\mathbf{w}) = \frac{1}{2\mu v}\left(\iiint_v \mathbf{k}\cdot\boldsymbol{\varepsilon}'\cdot\mathbf{r}\, d\tau_0 + \iint_O \mathbf{f}\cdot\boldsymbol{\varepsilon}'\cdot\mathbf{r}\, do\right) \tag{7.16}$$

in which $\boldsymbol{\varepsilon}'$ is some constant symmetric tensor, $\boldsymbol{\varepsilon}_m$ $(\mathbf{w})$ is the mean value of the strain tensor $\boldsymbol{\varepsilon}(\mathbf{w})$ relative to the volume v, and the ϑ', $\vartheta_m(\mathbf{w})$ denote the first invariants of the tensors $\boldsymbol{\varepsilon}'$, $\boldsymbol{\varepsilon}(\mathbf{w})$, and $\boldsymbol{\varepsilon}'\cdot\cdot\,\boldsymbol{\varepsilon}_m(\mathbf{w})$ the first invariant of their products; $\mathbf{r} = \mathbf{i}_s a_s$.

Referring to (7.13), (7.12), (7.10) we have

$$\iint_O \mathbf{f}\cdot\boldsymbol{\varepsilon}'\cdot\mathbf{r}\, do = -\iint_O [\boldsymbol{\omega}\times(\mathbf{n}\cdot\mathbf{T}^\circ) + \mathbf{n}\cdot\mathbf{T}^\bullet(\mathbf{v})]\cdot\boldsymbol{\varepsilon}'\cdot\mathbf{r}\, do =$$

$$= \iint_O \mathbf{n}\cdot(\mathbf{T}^\circ\times\boldsymbol{\omega} - \mathbf{T}^\bullet)\cdot\boldsymbol{\varepsilon}'\cdot\mathbf{r}\, do = \iint_O \mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{E}'\cdot\mathbf{r}\, do = \iiint_v (\nabla\cdot\mathbf{Q})\cdot\boldsymbol{\varepsilon}'\cdot\mathbf{r}\, d\tau_0 +$$

$$+ \iiint \mathbf{Q}\, d\tau_0\cdot\cdot\,\boldsymbol{\varepsilon}' = -\iiint_v \mathbf{k}\cdot\boldsymbol{\varepsilon}'\cdot\mathbf{r}\, d\tau_0 + \boldsymbol{\varepsilon}'\cdot\cdot\iiint_v \mathbf{Q}\, d\tau_0$$

and its substitution into (7.16) results in the desired relationship

$$\frac{\nu}{1-2\nu}\,\vartheta'\vartheta_m(\boldsymbol{\omega}) + \boldsymbol{\varepsilon}'\cdot\cdot\left[\boldsymbol{\varepsilon}_m - \frac{1}{2\mu v}\iiint_v (\mathbf{T}^\circ\times\boldsymbol{\omega} - \mathbf{T}^\bullet)\, d\tau_0\right] = 0 \tag{7.17}$$

Putting successively $\varepsilon' = E$, $\varepsilon' = i_1 i_1$, $\varepsilon' = i_1 i_2 + i_2 i_1$, we arrive at the expressions

$$\vartheta_m(\mathbf{w}) = -\frac{1-2\nu}{2\mu v(1+\nu)} \iiint_v I_1(T^\circ(\mathbf{v}))\, d\tau_0 \quad (7.18)$$

$$\frac{\nu}{1-2\nu}\vartheta_m(\mathbf{w}) + [\varepsilon_{11}(\mathbf{w})]_m = \frac{1}{2\mu v} \iiint_v [t_{12}^\circ(\mathbf{v})\,\omega_3 - t_{13}^\circ(\mathbf{v})\,\omega_2 - t_{11}^\circ(\mathbf{v})]\, d\tau_0 \quad (7.19)$$

$$2[\varepsilon_{12}(\mathbf{w})]_m = \frac{1}{2\mu v} \iiint_v (t_{22}^\circ(\mathbf{v}) - t_{11}^\circ(\mathbf{v}))\omega_3 - t_{23}^\circ(\mathbf{v})\,\omega_2 + t_{13}^\circ(\mathbf{v})\,\omega_1 - 2t_{12}^\circ(\mathbf{v})]\, d\tau_0 \quad (7.20)$$

It is easy to verify that the integrand in (7.20) is symmetric relative to the subscripts 1, 2 by replacing t_{sk}°, ω_r by their expressions in terms of derivatives of the displacement vector.

Example. Torsion of a rod. The solution is well known in a linear approximation

$$v_1 = -\alpha a_2 a_3,\quad v_2 = \alpha a_3 a_1,\quad v_3 = \alpha\varphi(a_1, a_2)$$

where φ is a harmonic function determined by the solution of the Neumann problem

$$\nabla^2\varphi = 0, \qquad \left.\frac{\partial\varphi}{\partial n}\right|_\Gamma = n_1 a_2 - n_2 a_1$$

The nonzero stresses

$$t_{31}^\circ = \mu\alpha\left(\frac{\partial\varphi}{\partial a_1} - a_2\right), \qquad t_{23}^\circ = \mu\alpha\left(\frac{\partial\varphi}{\partial a_2} + a_1\right)$$

in the expressions for the components of the linear rotation vector are written as

$$2\omega_1 = \alpha\left(\frac{\partial\varphi}{\partial a_2} - a_1\right), \qquad 2\omega_2 = -\alpha\left(\frac{\partial\varphi}{\partial a_1} + a_2\right), \quad \omega_3 = \alpha a_3$$

Let us verify compliance with the criterion (7.14). We have

$$\iint_S [\omega\cdot T^\circ(\mathbf{v}) - \omega I_1(T^\circ)]\, do = -\mu\alpha^2 \iint_S \left(a_1\frac{\partial\varphi}{\partial a_1} + a_2\frac{\partial\varphi}{\partial a_2}\right) do =$$

$$= \mu\alpha^2\left[2\iint_S \varphi\, do - \oint_\Gamma \varphi(a_1 n_1 + a_2 n_2)\, ds\right] = \oint_\Gamma \varphi\frac{\partial\varphi}{\partial s}\, ds = 0$$

since φ is single-valued in S. Here, we used the expression of the integral of a harmonic function $\psi(a_1, a_2)$ in a domain which is used in the problem of the center of stiffness, in terms of its contour value and the smoothness

$$2\iint \Psi(a_1, a_2)\, do - \oint_\Gamma \Psi(n_1 a_1 + n_2 a_2)\, ds = \oint_\Gamma \Psi\frac{\partial\varphi}{\partial s}\, ds$$

According to (7.4) and (7.18) we have

$$I_1(T^\circ) = (3\lambda + 2\mu)(\omega_1^2 + \omega_2^2 + \omega_3^2), \qquad t_{33}^\circ = \lambda\omega_3^2$$

$$\vartheta_m(\mathbf{w}) = -\frac{1-2\nu}{2\mu v(1+\nu)}(3\lambda + 2\mu)\int_0^l da_3 \iint_S (\omega_1^2 + \omega_2^2 + \omega_3^2)\, do$$

and upon substitution for $[\varepsilon_{33}(\mathbf{w})]_m$ in (7.19), the quantity ω_3 cancels. We obtain

$$(\varepsilon_{33})_m = \frac{1}{2S}\left[\frac{2\nu}{1-2\nu}\iint_S (\omega_1^2 + \omega_2^2)\, do + \frac{1}{\mu}\iint_S (t_{31}^\circ\omega_2 - t_{23}^\circ\omega_1)\, do\right] =$$

$$= \frac{\alpha^2}{4S(1-2\nu)}\left[(3\nu - 1)\iint_S (\nabla\varphi)^2 do + (1-\nu)\iint_S (a_1^2 + a_2^2)\, do + \right.$$

$$\left. + 2\nu\iint_S \left(a_2\frac{\partial\varphi}{\partial a_1} - a_1\frac{\partial\varphi}{\partial a_2}\right) do\right]$$

The integrals herein are expressed in terms of the torsion stiffness of the rod C and the polar moment of inertia I_p of the area

$$\iint_S (\nabla\varphi)^2 do = \iint_S \left(a_2 \frac{\partial\varphi}{\partial a_1} - a_1 \frac{\partial\varphi}{\partial a_2}\right) do = I_p - C; \quad I_p = \iint_S (a_1^2 + a_2^2)\, do$$

The change in rod length turns out to be

$$\Delta l = \frac{l\alpha^2}{4S(1-2\nu)} [4(I_p - C)\nu + C(1-\nu)]$$

and since $I_p \geqslant C$ (the equality is for a circular rod), it is positive for a material given by the specific potential energy (5.7), the rod elongates under torsion.

8. Superposition of a small strain upon a finite strain. Three states of a medium are examined: the natural (v), the stressed (V°) and the stressed (V) produced by communicating a displacement field given by the vector $\eta\mathbf{w}$

$$\mathbf{R} = \mathbf{R}^\circ + \eta\mathbf{w} \tag{8.1}$$

to points of the medium in the state V°.

The notation used earlier ($\mathbf{R}$, $\mathbf{D}$, $\mathbf{A}$, etc.) is retained for quantities in the V-state, their values in V^0 are distinguished by the small zero superscript on the right. The differences ("perturbations"), evaluated by keeping the first power of the smallness parameter η, are represented as the product of this parameter by a quantity denoted with a dot overhead

$$\dot{\mathbf{R}} = \mathbf{R}^\circ + \eta\dot{\mathbf{R}}, \qquad \mathbf{D} = \mathbf{D}^\circ + \eta\dot{\mathbf{D}}, \ \mathbf{A} = \mathbf{A}^\circ + \eta\dot{\mathbf{A}} \ \text{etc.}$$

Evidently $\dot{\mathbf{R}} = \mathbf{w}$, and the quantities with the dot are differential operators on the vector $\mathbf{w}$. They can be difined as derivatives of quantities in the V-state with respect to η at $\eta = 0$

$$\dot{\mathbf{D}} = \left(\frac{\partial\mathbf{D}}{\partial\eta}\right)_{\eta=0}, \qquad \dot{\mathbf{A}} = \left(\frac{\partial\mathbf{A}}{\partial\eta}\right)_{\eta=0} \quad \text{etc.}$$

In conformity with the definition (1.6)

$$\dot{\mathbf{G}}^\times = \nabla\mathbf{w}\cdot\nabla\mathbf{R}^{\circ T} + \nabla\mathbf{R}^\circ\cdot\nabla\mathbf{w}^T$$

The principal values G_s° of the tensors $\mathbf{G}^{\times\circ}$ and $\mathbf{g}^{\times(-1)\circ}$, and their principal directions $\mathbf{e}_s^\circ$, $\mathbf{e}_s'^\circ$ (see (1.9)) are assumed known. To construct

$$\dot{\mathbf{A}} = \dot{\mathbf{e}}_s\mathbf{e}_s'^\circ + \mathbf{e}_s^\circ\dot{\mathbf{e}}_s' \tag{8.2}$$

expressions must be formed for the vectors $\dot{\mathbf{e}}_s$, $\dot{\mathbf{e}}_s'$; in passing, the quantities $\dot{G}_s$ will also be found. These vectors are orthogonal to $\mathbf{e}_s^\circ$, $\mathbf{e}_s^{\circ\prime}$ since

$$\mathbf{e}_s\cdot\mathbf{e}_s = 1, \qquad \mathbf{e}_s'\cdot\mathbf{e}_s' = 1; \qquad \dot{\mathbf{e}}_s\cdot\mathbf{e}_s^\circ = 0 \qquad \dot{\mathbf{e}}_s'\cdot\mathbf{e}_s'^\circ = 0 \tag{8.3}$$

By the definition of the principal values and principal directions of the tensor

$$\mathbf{G}^\times\cdot\mathbf{e}_s - G_s\mathbf{e}_s = 0, \quad \mathbf{G}^{\times\circ}\cdot\dot{\mathbf{e}}_s - G_s^\circ\dot{\mathbf{e}}_s = \dot{G}_s\mathbf{e}_s^\circ - \dot{\mathbf{G}}^\times\cdot\mathbf{e}_s^\circ \quad (s = 1, 2, 3) \tag{8.4}$$

where we also have in the V^0 volume

$$\mathbf{G}^{\times\circ}\cdot\mathbf{e}_s^\circ - G_s^\circ\mathbf{e}_s^\circ = 0$$

Hence, multiplying (8.4) scalarly by $\mathbf{e}_k^\circ$, we obtain

$$(G_k^\circ - G_s^\circ)\,\mathbf{e}_k^\circ\cdot\dot{\mathbf{e}}_s = \dot{G}_s\delta_{sk} - \mathbf{e}_k^\circ\cdot\dot{\mathbf{G}}^\times\cdot\mathbf{e}_s^\circ$$

so that

$$\dot{G}_s = \mathbf{e}_s^\circ\cdot\dot{\mathbf{G}}^\times\cdot\mathbf{e}_s^\circ, \qquad \mathbf{e}_k^\circ\cdot\dot{\mathbf{e}}_s = \frac{\mathbf{e}_k^\circ\cdot\dot{\mathbf{G}}^\times\cdot\mathbf{e}_s^\circ}{G_s^\circ - G_k^\circ} \qquad (s, k = 1, 2, 3;\ s \neq k) \tag{8.5}$$

The projections of $\dot{\mathbf{e}}_s$ on axes orthogonal to the trihedron $\dot{\mathbf{e}}_k$ are defined by the last equality in combination with (8.4); hence, referring to (8.1)

$$\dot{e}_s = \sum_k{}' \frac{e_k^\circ \cdot \dot{G}^\times \cdot e_s^\circ}{G_s^\circ - G_k^\circ} e_k^\circ = \sum_k{}' \frac{1}{G_s^\circ - G_k^\circ} [e_k^\circ \cdot (\nabla w \cdot \nabla R^{\circ T} + \nabla R^\circ \cdot \nabla w^T) \cdot e_s^\circ e_k^\circ] \qquad (8.6)$$

Analogously we obtain

$$\dot{e}_k' = \sum{}' \frac{1}{G_k^\circ - G_s^\circ} [e_s'^\circ \cdot (\nabla w^T \cdot \nabla R^\circ + \nabla R^{\circ T} \cdot \nabla w) \cdot e_k'^\circ e_s'^\circ]$$

and the expression for the tensor $\dot{A}$ is written according to (8, 2) as (we discard the summation sign)

$$\dot{A} = \frac{e_k^\circ e_s'^\circ}{G_s^\circ - G_k^\circ} [e_k^\circ \cdot (\nabla w \cdot \nabla R^{\circ T} + \nabla R^\circ \cdot \nabla w^T) \cdot e_s^\circ - - e_s'^\circ \cdot (\nabla R^{\circ T} \cdot \nabla w + \nabla w^T \cdot \nabla R^\circ) \cdot e_k'^\circ]$$

Now, referring to (2, 6), (2, 1), (1, 10), (1, 13) and making substitutions of the form

$$e_k'^\circ \cdot \nabla w^T \cdot e_s^\circ = e_s^\circ \cdot \nabla w \cdot e_k'^\circ$$

we arrive after manipulation at

$$\dot{A} = \frac{e_k^\circ \cdot \nabla w \cdot e_s^{\circ\prime}}{\sqrt{G_s^\circ} + \sqrt{G_k^\circ}} (e_k^\circ e_s^{\circ\prime} - e_s^\circ e_k^{\circ\prime}) \qquad (8.7)$$

Furthermore, according to (8, 5) and (8, 1) we have

$$\dot{s}_1 = \dot{I}_1 (G^{\times 1/2}) = \left[\frac{\partial}{\partial \eta} I_1 (G^{\times 1/2})\right]_{\eta=0} = \frac{\partial}{\partial \eta}\left(\sum_k \sqrt{G_k}\right)_{\eta=0} = \frac{1}{2} \sum_k \frac{\dot{G}_k}{\sqrt{G_k^\circ}} =$$

$$= \frac{1}{2} \sum_k \frac{1}{\sqrt{G_k^\circ}} e_k^\circ \cdot \dot{G}^\times \cdot e_k^\circ = \frac{1}{2} \sum_k [e_k^\circ \cdot \nabla w \cdot e_k'^\circ + e_k'^\circ \cdot \nabla w^T \cdot e_k^\circ]$$

so that

$$\dot{s}_1 = \sum_k e_k^\circ \cdot \nabla w \cdot e_k'^\circ \qquad (8.8)$$

and now according to (5, 9)

$$\dot{D} = \frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ} + \sqrt{G_k^\circ}} e_k^\circ \cdot \nabla w \cdot e_s'^\circ (e_k^\circ e_s'^\circ - e_s^\circ e_k'^\circ) + \lambda A^\circ e_k^\circ \cdot \nabla w \cdot e_k'^\circ + 2\mu \nabla w \qquad (8.9)$$

The equilibrium equations in the volume (in the absence of mass forces) and on the surface are written as

$$\nabla \cdot \dot{D} = 0 \text{ in } v; \qquad (F\, dO)^\cdot = n \cdot \dot{D} do \quad \text{on } o \qquad (8.10)$$

In the particular case when the surface force F is a constant pressure p, which remains normal to the surface O, we have by (3, 2), (1, 2)

$$F = -pN, \qquad (N dO)^\cdot = (\sqrt{G/g}\, R^s)^\cdot n_s do$$

and from the relationship

$$(\sqrt{G/g})^\cdot = \sqrt{G^\circ/g} \sum_k \frac{e_k^\circ \cdot \nabla w \cdot e_k'^\circ}{\sqrt{G_k^\circ}}, \quad \dot{R}^s = -R^s \cdot \dot{R}_t R^{\circ t}$$

we obtain

$$n \cdot \dot{D} = -p \sqrt{\frac{G^\circ}{g}} \left[\sum_k \frac{e_k^\circ \cdot \nabla w \cdot e_k'^\circ}{\sqrt{G_k^\circ}} R^{\circ s} - R^{\circ s} \cdot \frac{\partial w}{\partial q^t} R^{\circ t}\right] n_s \qquad (8.11)$$

9. Conservation of the principal directions. In this case (Sect. 6), Eqs. (8, 9)–(8, 11) simplify considerably. The tensor ∇ w is represented by its partition into symmetric and antisymmetric parts

$$\nabla w = \varepsilon - \Omega = \varepsilon - E \times \omega \qquad (9.1)$$

where ε is the linear strain tensor, ω the linear rotation vector evaluated by means of the displacement vector w. Also taking account of (6, 1), (6, 2), we obtain an equation reducible to another Sensenig [2] form

$$\dot{D} = 2 \frac{\lambda s_1^\circ - 2\mu}{\sqrt{G_s^\circ} + \sqrt{G_k^\circ}} \omega \cdot (e_k^\circ \times e_s^\circ) e_k^\circ e_s^\circ + \lambda E \nabla \cdot w + 2\mu (\varepsilon - E \times \omega) \qquad (9.2)$$

But it is easy to verify that

$$-\mathbf{E}\times\boldsymbol{\omega} = -\mathbf{e}_k{}^\circ\mathbf{e}_k{}^\circ\times\mathbf{e}_r{}^\circ\omega_r = \mathbf{e}_k{}^\circ\mathbf{e}_s{}^\circ \in_{ksr}\omega_r = \mathbf{e}_k{}^\circ\mathbf{e}_s{}^\circ(\mathbf{e}_k{}^\circ\times\mathbf{e}_s{}^\circ)\cdot\boldsymbol{\omega}$$

where $\in_{ksr} = (\mathbf{e}_k{}^\circ\times\mathbf{e}_s{}^\circ)\cdot\mathbf{e}_r{}^\circ$ is the Levi-Civita symbol. This permits writing (9.2) as

$$\dot{\mathrm{D}} = \mathrm{T}(\mathbf{w}) - 2\left(\frac{\lambda s_1{}^\circ - 2\mu}{\sqrt{G_s{}^\circ}+\sqrt{G_k{}^\circ}} + \mu\right)\mathbf{E}\times\boldsymbol{\omega} \tag{9.3}$$

where $\mathrm{T}(\mathbf{w})$ is the linear stress tensor evaluated by means of the vector $\mathbf{w}$

$$\mathrm{T}(\mathbf{w}) = \mathbf{E}\lambda\nabla\cdot\mathbf{w} + 2\mu\boldsymbol{\varepsilon} \tag{9.4}$$

According to (6.1), (6.2) and (5.8), the tensor D° is represented in the volume V° as

$$\mathrm{D}^\circ = (\lambda s_1{}^\circ - 2\mu)\,\mathbf{E} + 2\mu\sum\sqrt{G_s{}^\circ}\,\mathbf{e}_s{}^\circ\mathbf{e}_s{}^\circ = \lambda s_1{}^\circ\mathbf{E} + 2\mu\sum_s \delta_s{}^\circ\mathbf{e}_s{}^\circ\mathbf{e}_s{}^\circ \quad (\sqrt{G_s{}^\circ} = 1 + \delta_s{}^\circ) \tag{9.5}$$

where $\delta_s{}^\circ$ are the principal elongations (see (1.10)). This permits transformation of the expressions in (9.2) to

$$2\left(\frac{\lambda s_1{}^\circ - 2\mu}{\sqrt{G_1{}^\circ}+\sqrt{G_2{}^\circ}} + \mu\right) = 2\,\frac{\lambda s_1{}^\circ + \mu(\delta_1{}^\circ + \delta_2{}^\circ)}{2+\delta_1{}^\circ+\delta_2{}^\circ} = \frac{\partial_1{}^\circ + \partial_2{}^\circ}{2+\delta_1{}^\circ+\delta_2{}^\circ} \quad \text{etc.}$$

where ∂_s° is a component of the tensor D°. Now, defining the diagonal tensor C in the $\mathbf{e}_s{}^\circ$ axes

$$\mathrm{C} = \sum_s C_s\mathbf{e}_s{}^\circ\mathbf{e}_s{}^\circ,\quad C_1 = \frac{1}{2\mu}\frac{\partial_2{}^\circ + \partial_3{}^\circ}{2+\delta_2{}^\circ+\delta_3{}^\circ},\quad C_2 = \frac{1}{2\mu}\frac{\partial_3{}^\circ + \partial_1{}^\circ}{2+\delta_3{}^\circ+\delta_1{}^\circ}$$

$$C_3 = \frac{1}{2\mu}\frac{\partial_1{}^\circ + \partial_2{}^\circ}{2+\delta_1{}^\circ+\delta_2{}^\circ} \tag{9.6}$$

the expression for D can be rewritten in the invariant form

$$\mathrm{D} = \mathrm{T}(\mathbf{w}) - 2\mu\mathbf{E}\times(\mathrm{C}\cdot\boldsymbol{\omega}) \tag{9.7}$$

Let us note that

$$\frac{1}{\mu}\nabla\cdot\mathrm{T}(\mathbf{w}) = \frac{1}{1-2\nu}\nabla\vartheta + \nabla^2\mathbf{w} = \frac{2(1-\nu)}{1-2\nu}\nabla\vartheta - 2\nabla\times\boldsymbol{\omega} \quad (\vartheta = \nabla\cdot\mathbf{w})$$

and the equilibrium equation (8.10) can be represented as

$$\nabla\vartheta - \frac{1-2\nu}{2(1-\nu)}2\nabla\times(\mathrm{C}+\mathrm{E})\cdot\boldsymbol{\omega} = 0, \qquad \nabla\vartheta - 2\nabla\times\mathrm{B}\cdot\boldsymbol{\omega} = 0 \tag{9.8}$$

where we have introduced the tensor

$$\mathrm{B} = \frac{1-2\nu}{2(1-\nu)}(\mathrm{C}+\mathrm{E}) \tag{9.9}$$

Such are the equilibrium equations in displacements for a "semilinear" material when the initial stresses therein are subject to conditions of conserving the principal directions (Sect. 6).

10. Homogeneous strain case. Under homogeneous strain

$$x_s = a_s(1+\delta_s{}^\circ), \qquad \mathbf{e}_s{}^\circ = \mathbf{i}_s \tag{10.1}$$

where $\delta_s{}^\circ$ means the tensors B, C are constant. In this case (9.8) reduces to the "neutral equilibrium equations" form of Southwell [3]. In these latter the quantities $\partial_s{}^\circ$ are identified with the principal stresses $\sigma_s{}^\circ$ in the volume V°; this is untrue, the correct relationships should be written as

$$\sigma_s{}^\circ = \partial_s{}^\circ\frac{1+\delta_s{}^\circ}{(1+\delta_1{}^\circ)(1+\delta_2{}^\circ)(1+\delta_3{}^\circ)} \qquad (s = 1, 2, 3) \tag{10.2}$$

But since it is assumed that the stress tensor T° is connected with the relative elongations $\delta_s{}^\circ$ by means of the relationship (9.5), the Southwell equations are true if the constants C_s therein are expressed in terms of $\delta_s{}^\circ$

The general solution of (9.8) under homogeneous strain can be represented in terms of the vector $\mathbf{G}$ thus

$$\mathbf{w} = (D_1^2 + D_2^2)\nabla\nabla\cdot\mathbf{G} - (\mathbf{B}\cdot\nabla)\nabla\cdot\nabla^2\mathbf{G} - \nabla\times[\nabla\times(\mathbf{B}\cdot\nabla^2\mathbf{G})] \tag{10.3}$$

Here D_1^2, D_2^2 are differential operators

$$D_1^2 = B_1\partial_1^2 + B_2\partial_2^2 + B_3\partial_3^2 = (\mathbf{B}\cdot\nabla)\cdot\nabla \qquad \left(\partial_s = \frac{\partial}{\partial a_s}\right)$$

$$D_2^2 = B_1B_2B_3\left(\frac{\partial_1^2}{B_1} + \frac{\partial_2^2}{B_2} + \frac{\partial_3^2}{B_3}\right) = B(\mathbf{B}^{-1}\cdot\nabla)\cdot\nabla \qquad (B = B_1B_2B_3) \tag{10.4}$$

and the vector $\mathbf{G}$ is defined by the differential equation

$$\nabla^4 D_2^2\mathbf{G} = 0 \tag{10.5}$$

In particular, under multilateral uniform compression $\delta_s^\circ = \delta^\circ$, we will have $\mathbf{B} = B_0\mathbf{E}$. Introducing the new vector $\mathbf{G}^* = -B_0\nabla^2\mathbf{G}$, we obtain

$$\mathbf{w} = -B_0\nabla\nabla\cdot\mathbf{G}^* + \nabla\times(\nabla\times\mathbf{G}^*) = (1 - B_0)\nabla\nabla\ \mathbf{G}^* - \nabla^2\mathbf{G}^* \tag{10.6}$$

where $\mathbf{G}^*$ is a biharmonic vector. Here, according to (9.5), (9.9)

$$1 - B_0 = \frac{1}{2(1-\nu)}\left[1 - (1+\nu)\frac{\delta_0}{2(1+\delta_0)}\right] \tag{10.7}$$

For $\delta_0 = 0$ we return to the known Boussinesq-Galerkin solution.

The solution (10.3) can be simplified if the vector $\mathbf{G}$ is represented as the sum of the vectors $\mathbf{G}'$ and $\mathbf{G}''$ expressed in terms of a scalar and a solenoidal vector

$$\mathbf{G}' = \mathbf{B}^{-1}\cdot\nabla\varphi, \qquad \nabla^2\mathbf{G}'' = \mathbf{H}, \qquad \nabla\cdot\mathbf{G}'' = 0, \qquad \nabla\cdot\mathbf{H} = 0$$

Then

$$\nabla\cdot\mathbf{G}' = \frac{D_2^2\varphi}{B_1B_2B_3} = \Phi, \quad \nabla\cdot\nabla^2\mathbf{G}' = \nabla^2\Phi, \quad \nabla\times[\nabla\times(\mathbf{B}\cdot\nabla^2\mathbf{G}')] = 0$$

where the scalar Φ is biharmonic according to (10.5). Its corresponding solution is represented as

$$\mathbf{w}' = (D_1^2 + D_2^2)\nabla\Phi - (\mathbf{B}\cdot\nabla)\nabla^2\Phi \tag{10.8}$$

The second solution is determined by means of the vector $\mathbf{H}$

$$\mathbf{w}'' = -\nabla\times[\nabla\times(\mathbf{B}\cdot\mathbf{H})] = \nabla^2\mathbf{B}\cdot\mathbf{H} - \nabla\nabla\cdot(\mathbf{B}\cdot\mathbf{H}) \tag{10.9}$$

where the vector $\mathbf{H}$ is defined by equations for $\mathbf{H}$

$$\nabla^2 D_2^2\mathbf{H} = 0, \qquad \nabla\cdot\mathbf{H} = 0 \tag{10.10}$$

11. A compressed rod. A vertical rod is located between two solid, smooth slabs; its lateral surface is not loaded. A uniaxial state of stress is produced by a downward vertical shift of the upper slab ($a_3 = L$) in the amount of $L\delta_3^\circ$ while the lower slab ($a_3 = 0$) is fixed. In this state $\partial_1^\circ = \partial_2^\circ = 0$, and according to (9.5), (9.6)

$$\delta_3^\circ = \frac{\partial_3^\circ}{E} \qquad \delta_1^\circ = \delta_2^\circ = -\nu\delta_3^\circ, \qquad C_1 = C_2 = C = \frac{(1+\nu)\delta_3^\circ}{2 + (1-\nu)\delta_3^\circ}, \qquad C_3 = 0$$

so that

$$B_3 = \alpha = \frac{1-2\nu}{2(1-\nu)}, \qquad B_1 = B_2 = \alpha\sigma$$

$$\sigma = \frac{2(1+\delta_3^\circ)}{2 + (1-\nu)\delta_3^\circ}, \qquad \delta_3^\circ = -\frac{2(1-\sigma)}{2-(1-\nu)\sigma} \tag{11.1}$$

A parameter σ has been introduced here, where $0 < \sigma < 1$ since $-1 < \delta_3^\circ < 0$.

The three boundary conditions on the remaining unloaded lateral surface are represented as

$$\mathbf{n}\cdot\mathbf{D}' = \mathbf{n}\cdot\mathbf{T}(\mathbf{w}) + 2\mu C(\omega_1 n_2 - \omega_2 n_1)\mathbf{i}_3 = 0 \tag{11.2}$$

On the rod endfaces

$$\mathbf{n}\cdot\dot{\mathbf{D}} = \mathbf{i}_3\cdot\dot{\mathbf{D}} = \mathbf{i}_3\cdot\mathbf{T}(\mathbf{w}) - 2\mu C(\omega_1\mathbf{i}_2 - \omega_2\mathbf{i}_1) \tag{11.3}$$

and the horizontal projections of this force and the vertical displacement on the endfaces should be taken equal to zero $\mathbf{i}_3\cdot\mathbf{D}\cdot\mathbf{i}_1=0, \quad \mathbf{i}_3\cdot\mathbf{D}\cdot\mathbf{i}_2=0, \quad \mathbf{w}=0$

in seeking the equilibrium mode different from the homogeneous state of strain, but realized by the method described above.

This results in the conditions (u, v, w are projections of the vector $\mathbf{w}$)

$$\partial_3 u = 0, \quad \partial_3 v = 0, \quad w = 0 \quad \text{for } a_3 = 0,\ a_3 = L \tag{11.4}$$

which are automatically satisfied if it is assumed that u, v are proportional to the cosine, and w to the sine of the argument $(n\pi/L)\,a_3$.

We take

$$\Phi = \varphi(a_1, a_2)\cos\frac{n\pi}{L}a_3$$

$$H_1 = \partial_2\Psi, \quad H_2 = -\partial_1\psi, \quad H_3 = 0; \quad \nabla^2\Psi = \frac{n^2\pi^2}{L^2\sigma}\Psi(a_1, a_2)\cos\frac{n\pi}{L}a_3$$

in the solutions (10.8)–(10.10), where the condition that $\mathbf{H}$ is a solinoidal vector is satisfied, and the functions $\varphi_n(a_1, a_2)$ and $\psi_n(a_1, a_2)$ are determined by the differential equations

$$\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right)^2\varphi(a_1, a_2) = 0, \qquad \left(\nabla_1^2 - \sigma\frac{n^2\pi^2}{L^2}\right)\psi_n(a_1, a_2) = 0$$
$$(\nabla_1^2 = \partial_1^2 + \partial_2^2) \tag{11.5}$$

Expressions of the displacements (10.8), (19.9) are written as

$$u = \left\{\left[\alpha\sigma\nabla_1^2 - \frac{n^2\pi^2}{L^2}(1-\sigma+\alpha\sigma^2)\right]\partial_1\varphi_n + \frac{n^2\pi^2}{L^2}\partial_2\psi_n\right\}\cos\frac{n\pi}{L}a_3$$
$$v = \left\{\alpha\sigma\nabla_1^2 - \frac{n^2\pi^2}{L^2}(1-\sigma+\alpha\sigma^2)\right]\partial_2\varphi_n - \frac{n^2\pi^2}{L^2}\partial_1\psi_n\right\}\cos\frac{n\pi}{L}a_3 \tag{11.6}$$
$$w = \left[(1-\sigma-\alpha\sigma)\nabla_1^2 + \alpha\sigma^2\frac{n^2\pi^2}{L^2}\right]\frac{n\pi}{L}\varphi_n\sin\frac{n\pi}{L}a_3$$

and the volume expansion ϑ evaluated therewith is represented by the equality

$$\vartheta = \alpha\sigma\frac{n^2\pi^2}{L^2}(1-\sigma)\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right)\varphi_n\cos\frac{n\pi}{L}a_3$$

The boundary conditions (11.3) reduce to

$$n_1\frac{\nu}{1-2\nu}\sigma(1-\sigma)\frac{n^2\pi^2}{L^2}\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right)\varphi_n + \frac{\partial}{\partial n}\left[\alpha\sigma\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right) - \right.$$
$$\left. - \frac{n^2\pi^2}{L^2}(1-\sigma)(1-\alpha\sigma)\right]\partial_1\varphi_n + \frac{n^2\pi^2}{2L^2}\left(\frac{\partial}{\partial n}\partial_2\psi + \frac{\partial}{\partial s}\partial_1\psi\right) = 0$$

$$n_2\frac{\nu}{1-2\nu}\sigma(1-\sigma)\frac{n^2\pi^2}{L^2}\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right)\varphi_n + \frac{\partial}{\partial n}\left[\alpha\sigma\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right) - \right. \tag{11.7}$$
$$\left. - \frac{n^2\pi^2}{L^2}(1-\sigma)(1-\alpha\sigma)\right]\partial_2\varphi_n + \frac{n^2\pi^2}{2L^2}\left(\frac{\partial}{\partial s}\partial_2\psi - \frac{\partial}{\partial n}\partial_1\psi\right) = 0$$

$$\frac{\partial}{\partial n}\left[(\sigma - 2\alpha\sigma - \sigma^2)\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right) + 2(1-\sigma)(1-\alpha\sigma)\frac{n^2\pi^2}{L^2}\right]\varphi_n - \frac{n^2\pi^2}{L^2}(2-\sigma)\frac{\partial\psi_n}{\partial s} = 0$$

Values of the parameter σ in the interval (0, 1) for which the homogeneous boundary value problem (11.5), (11.7) has a nontrivial solution, will be bifurcation values. Besides the original equilibrium mode ($\mathbf{w} = 0$), nearby modes also exist for such σ.

12. Rod of circular cross section [2]. Solutions of (11.5) are sought in the form $\varphi_n(a_1, a_2) = R(x)\cos m\theta, \ \psi_n(a_1, a_2) = S(x\sqrt{\sigma})\sin m\theta \ (x = n\pi r/L)$

where $R(x)$, $S(x\sqrt{\sigma})$ are expressed in terms of Bessel functions, and the variable θ is

$$R = R_0: \left[\lambda + p_0 \frac{f(R)}{R}\right] \nabla\cdot\mathbf{w} + \left[2\mu - p_0 \frac{f(R)}{R}\right] \frac{\partial w_R}{\partial R} = 0$$

$$2g(R)\,\omega_\lambda + \left(2\mu - p_0 \frac{f(R)}{R}\right)\left(\frac{\partial w_R}{\partial\vartheta} - w_\vartheta\right) = 0 \qquad (13.2)$$

$$R = R_1: \lambda\nabla\cdot\mathbf{w} + 2\mu\frac{\partial w_R}{\partial R} = 0, \qquad g(R)\,\omega_\lambda + \mu\left(\frac{\partial w_R}{\partial\vartheta} - w_\vartheta\right) = 0 \qquad (13.3)$$

The condition for the existence of nontrivial solutions of this homogeneous boundary value problem is determined by the bifurcation values of the pressure p_0.

The finite solution at the poles of the sphere is sought in the form

$$w_R = a_n(R)\,P_n(\cos\vartheta), \qquad w_\vartheta = b_n(R)\,\frac{dP_n(\cos\vartheta)}{d\cos\vartheta}\sin\vartheta$$

and utilizing the known properties of Legendre polynomials, we obtain

$$\nabla\cdot\mathbf{w} = \left[a_n' + 2\frac{a_n}{R} - n(n+1)\frac{b_n}{R}\right] P_n(\cos\vartheta) = \varphi_n(R)\,P_n(\cos\vartheta)$$

$$2\omega_\lambda = \left(b_n' + \frac{a_n + b_n}{R}\right)\frac{dP_n(\cos\vartheta)}{d\cos\vartheta}\sin\vartheta = \chi_n(R)\,\frac{dP_n(\cos\vartheta)}{d\cos\vartheta}\sin\vartheta \qquad (13.4)$$

The variables R, ϑ are separated in the equilibrium equations (13.1), whereupon a system of linear differential equations results

$$(\lambda + 2\mu)\,R^2\varphi_n' - Rg(R)\,n(n+1)\,\chi_n(R) = 0$$

$$(\lambda + \mu)\;\varphi_n(R) - [Rg(R)\,\chi_n(R)]' = 0$$

whose general solution is written thus

$$\varphi_n(R) = (n+1)\,A_nR^n - \frac{nB_n}{R^{n+1}}, \qquad \chi_n(R) = \frac{\lambda + 2\mu}{g(R)}\left(A_nR^n + \frac{B_n}{R^{n+1}}\right) \qquad (13.5)$$

Here, replacing φ_n, χ_n by their expressions in (13.4), we find after still another integration

$$a_n(R) = -nC_n^*R^{n-1} + (n+1)\frac{D_n^*}{R^{n+2}}, \qquad b_n(R) = C_n^*R^{n-1} + \frac{D_n^*}{R^{n+2}} \qquad (13.6)$$

where

$$C_n^* = C_n + \frac{1}{2n+1}\int_{R_1}^{R} [(n+1)\,\chi_n - \varphi_n]\,\frac{dR}{R^{n-1}}$$

$$D_n^* = D_n + \frac{1}{2n+1}\int_{R_1}^{R} (\varphi_n + n\chi_n)\,R^{n+2}dR \qquad (13.7)$$

The variables R and ϑ are also separated in the linear boundary conditions (13.2), (13.3); substituting the values found for a_n, b_n therein, we obtain a linear homogeneous system of four equations for the constants A_n, B_n, C_n, D_n. Bifurcation values of p_0 are determined by the condition that its determinant vanishes.

The problem of bifurcation of the axisymmetric equilibrium mode of a hollow circular cylinder compressed uniformly by distributed external pressure [2] is considered analogously.

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ASYMPTOTIC ANALYSIS OF SOME PLANE PROBLEMS OF THE THEORY OF ELASTICITY WITH COUPLE STRESSES

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The problem of stress concentration around curved holes without angular points, and the problem of an oscillating loading applied to the boundary of a half-plane are considered. The purpose is to investigate those properties of their solutions which result form smallness of the parameter l. The system is reduced to one equation, and an asymptotic method in the version of Vishik and Liusternik [1] is applied to solve it. For the concentration problem it is shown that if the solution by customary theory is known, then the solution by couple-stress theory can easily be constructed in a first approximation, and that couple-stress theory yields only an insignificant refinement. In the half-plane problem it is shown that the correction to the corresponding classical problem will be essential only in the case of rapid oscillation of the boundary conditions, i.e. when the state of stress being studied is of edge character. Another version of the asymptotic analysis of nonclassical problems of elasticity theory is given for fibrous media in [2].

1. In a Cartesian coordinate system the plane strain relationship in couple-stress elasticity theory, as presented in [3], are:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \quad \frac{\partial \mu_x}{\partial x}+\frac{\partial \mu_y}{\partial y}+\tau_{xy}-\tau_{yx}=0 \qquad (1.1)$$

$$\varepsilon_x=\frac{1+\nu}{E}\,[\sigma_x-\nu(\sigma_x+\sigma_y)], \qquad \varepsilon_y=\frac{1+\nu}{E}\,[\sigma_y-\nu(\sigma_x+\sigma_y)]$$

$$\varepsilon_{xy}=\frac{1+\nu}{2E}\,(\tau_{xy}+\tau_{yx}), \quad \chi_x=\frac{1}{4Gl^2}\mu_x, \quad \chi_y=\frac{1}{4Gl^2}\mu_y$$

$$\varepsilon_x=\frac{\partial u}{\partial x}, \quad \varepsilon_y=\frac{\partial v}{\partial y}, \quad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\chi_x=\frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\,\partial y}\right), \qquad \chi_y=\frac{1}{2}\left(\frac{\partial^2 v}{\partial x\,\partial y}-\frac{\partial^2 u}{\partial y^2}\right)$$

Here $\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx}$ and μ_x, μ_y are components of the force and moment stress tensors $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ and χ_x, χ_y are components of the strain and bending-torsion tensors; u, v the components of the displacement vector; E the Young's modulus, G the shear modulus; ν the Poisson coefficient; l the characteristic length of the material which we shall henceforth consider small as compared with the minimum radius of curvature of the hole.

Utilizing (1.1), we obtain two equations in the displacements